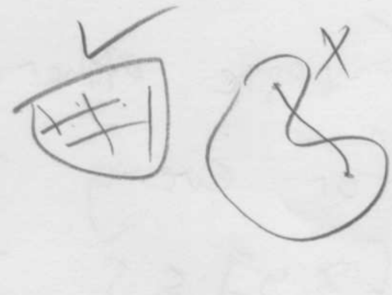
What is a convex set?

- **Defn** A line segment with endpoints $x, y \in \mathbb{R}^n$ is the set $$[x,y] := \{\lambda x + (1-\lambda)y \mid 0 \le \lambda \le 1\}.$$
- **Defn** A set $S \subseteq \mathbb{R}^n$ is convex if $$x, y \in S \Rightarrow [x,y] \subseteq S.$$



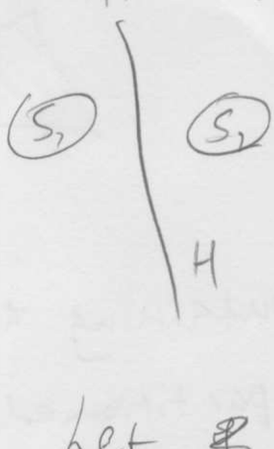
What makes convex sets cool?

Theorem (Separation)

Let $S_1, S_2 \subseteq \mathbb{R}^n$. Then there exists a hyperplane H in $\mathbb{R}^n$ that (properly) seperates S_1 & S_2 if:

(a) S_1 & S_2 are convex

(b) $\mathrm{ri}(S_1) \cap \mathrm{ri}(S_2) = \emptyset$.



Claim. Let S be a collection of finitely many points in the plane with pairwise distance no greater than 1. Then S is contained in a closed ball of radius no greater than $\frac{1}{\sqrt{3}}$.

Theory.

Theorem. Let $\{S_i\}_{i\in I}$ be a family of convex sets in $\mathbb{R}^n$. Then $\bigcap_{i\in I} S_i$ is convex

Proof. If $\left|\bigcap_{i\in I} S_i\right| \leq 1$ then we are done. Suppose otherwise & let $x, y \in \bigcap_{i\in I} S_i$. Then for every $i \in I$, $x, y \in S_i$ and hence $[x,y] \subseteq S_i$. Then $[x,y] \subseteq \bigcap_{i\in I} S_i$, so we have shown $\bigcap_{i\in I} S_i$ is convex. $\square$

Defn. Let $S \subseteq \mathbb{R}^n$. Then the convex hull of S, $co(S)$, is the smallest convex set in $\mathbb{R}^n$ that contains S.

Important results

Theorem (Radon). Let $S \subseteq \mathbb{R}^n$, containing at least $n+2$ points. Then S can be partitioned into two disjoint subsets S_1, S_2 s.t. $co(S_1) \cap co(S_2) \neq \emptyset$.

Theorem (Helly-finite).

Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be a family of $m \geq n+1$ convex sets in $\mathbb{R}^n$. If every ~~any~~ subfamily of $n+1$ sets in $\mathcal{F}$ has nonempty intersection, then $\bigcap \mathcal{F} \neq \emptyset$.

Proof. We induct on m.

If $m = n+1$ then there is nothing to prove.

Suppose $m > n+1$, & suppose that the result holds for all such families of fewer than m sets.

Then it follows that

$\forall i \in \{1, \ldots, m\} \ \exists x_i \in S_1 \cap \cdots \cap S_{i-1} \cap S_{i+1} \cap \cdots \cap S_m.$

Then we apply Radon's theorem to the set

$S = \{x_1, \ldots, x_m\}$, yielding two disjoint subsets $C_1, C_2 \subseteq S$. Then wlog let

$C_1 = \{x_1, \ldots, x_k\}, \quad C_2 = \{x_{k+1}, \ldots, x_m\}$

for some $k \in \{1, \ldots, m-1\}$.

Then note that

$\forall i \in \{1, \dots, k\}$: $x_i \in S_{k+1} \cap \dots \cap S_m$, and
$\forall i \in \{k+1, \dots, m\}$: $x_i \in S_1 \cap \dots \cap S_k$.

Given this, observe that

$co(x_1, \dots, x_k) \subseteq S_{k+1} \cap \dots \cap S_m$, and
$co(x_{k+1}, \dots, x_m) \subseteq S_1 \cap \dots \cap S_k$.

Now:

$$\begin{aligned} \bigcap \mathcal{F} &= S_1 \cap \dots \cap S_k \cap S_{k+1} \cap \dots \cap S_m \\ &\supseteq co(x_{k+1}, \dots, x_m) \cap co(x_1, \dots, x_k) \\ &= co(C_2) \cap co(C_1) \\ &\neq \emptyset. \end{aligned}$$

□

<u>Lemma</u>. Let $X \subseteq \mathbb{R}^n$. Then X is compact ~~iff~~ any family of closed subsets of X ~~has~~ with the finite intersection property has nonempty intersection.

<u>Theorem</u> (Helly – infinite). Let $\mathcal{F} = \{S_i\}_{i \in I}$ be a family of compact convex subsets in $\mathbb{R}^n$. If every subfamily of $n+1$ sets in $\mathcal{F}$ has nonempty intersection, then $\bigcap \mathcal{F} \neq \emptyset$.

Proof. Choose arbitrarily some $k \in I$ s.t. $S_k \in \mathcal{F}$.

Consider the family $\mathcal{G} := \mathcal{F} \cap S_k = \{ S_i \cap S_k \mid i \in I \}$.

Now let $\mathcal{H} \subseteq \mathcal{G}$ be a finite subfamily. Then there exists some finite $J \subseteq I$ s.t. $\mathcal{H} = \{ S_j \cap S_k \mid j \in J \}$. Then by Helly's theorem,

$$\begin{aligned}
\bigcap \mathcal{H} &= \bigcap \{ S_j \cap S_k \mid j \in J \} \\
&= \bigcap \{ S_j \mid j \in J \cup \{k\} \} \\
&\neq \emptyset.
\end{aligned}$$

Then $\mathcal{G}$ is a family of closed (by H-B) subsets of S_k w/ the FIP, & therefore by the Lemma, $\bigcap \mathcal{G} \neq \emptyset$.

Then notice

$$\begin{aligned}
\bigcap \mathcal{F} &= \left(\bigcap \mathcal{F} \right) \cap S_k \\
&= \bigcap (\mathcal{F} \cap S_k) \\
&= \bigcap \mathcal{G} \\
&\neq \emptyset.
\end{aligned}$$

□

Theorem (Jung).

Let $S \subseteq \mathbb{R}^n$. Then S is contained in a closed ball of radius $r \leq d\sqrt{\frac{n}{2(n+1)}}$, where $d = \mathrm{diam}(S) = \sup_{x,y \in S} |x-y|$.

Proof. $|S| \leq n+1$ ———

$|S| > n+1$. Let $\rho = d\sqrt{\frac{n}{2(n+1)}}$ and consider for each $x \in S$ the closed ball

$$\overline{B}_\rho(x). \qquad (+)$$

Then the result holds for ~~some~~ any $S' \subseteq S$ of $n+1$ or fewer elements of S, and therefore note that

$$S' \subseteq \overline{B}_\rho(z) \Leftrightarrow z \in \bigcap_{x \in S'} \overline{B}_\rho(x)$$

for some $z \in \mathbb{R}^n$.

Then apply Helly's theorem to the family $\mathcal{F} = \{ \overline{B}_\rho(x) \mid x \in S \}$, to show that $\bigcap_{x \in S} \overline{B}_\rho(x) \neq \emptyset. \Leftrightarrow \exists z \in \mathbb{R}^n : S \subseteq \overline{B}_\rho(z)$.

□