Convex sets in Euclidean space

What is a convex set?

Definition. A *line segment* with endpoints $x, y \in \mathbb{R}^n$ is the set

$$[x, y] := \{\lambda x + (1 - \lambda)y \mid 0 \leq \lambda \leq 1\}.$$

Definition. A set $S \subseteq \mathbb{R}^n$ is *convex* if

$$x, y \in S \implies [x, y] \subseteq S.$$

What makes convex sets cool?

Theorem (Separation). *Let $S_1, S_2 \subseteq \mathbb{R}^n$. There exists a hyperplane H in $\mathbb{R}^n$ that (properly) separates S_1 and S_2 if*

(a) S_1 and S_2 are convex, and

(b) $\mathrm{ri}(S_1) \cap \mathrm{ri}(S_2) \neq \emptyset$.

Claim. *Let $S \subseteq \mathbb{R}^2$ be a set of finitely many points in the plane, with pairwise distance no greater than 1. Then S is contained in a closed ball of radius no greater than $1/\sqrt{3}$.*

Some important principles

Theorem. *Let $\{S_i\}_{i \in \mathcal{I}}$ be a family of convex sets in $\mathbb{R}^n$. Then the intersection $\bigcap_{i \in \mathcal{I}} S_i$ is convex.*

Proof. If the intersection is empty, or contains exactly one point, then the claim follows immediately from the definition. Suppose otherwise, and let $x, y \in \bigcap_{i \in \mathcal{I}}$. Then for every $i \in \mathcal{I}$, we have $x, y \in S_i$ and hence $[x, y] \subseteq S_i$ by the convexity of S_i. Then clearly $[x, y] \subseteq \bigcap_{i \in \mathcal{I}}$, whence the result. □

Definition. Let $S \subseteq \mathbb{R}^n$. The *convex hull* of S, denoted by $\mathrm{co}(S)$, is the smallest convex set in $\mathbb{R}^n$ that contains S.

Important results

Theorem (Radon). *Let $S \subseteq \mathbb{R}^n$ contain at least $n+2$ elements. Then S can be partitioned into two disjoint subsets C_1 and C_2, such that* $\mathrm{co}(C_1) \cap \mathrm{co}(C_2) \neq \emptyset$.

Theorem (Helly, finite families). *Let $\mathscr{F} = \{S_1, \ldots, S_m\}$ be a family of $m \geq n + 1$ convex sets in $\mathbb{R}^n$. If every subfamily of $n + 1$ sets in $\mathscr{F}$ has nonempty intersection, then $\bigcap \mathscr{F} \neq \emptyset$.*

Proof. By induction on m.

If $m = n+1$, then there is nothing to prove.

Suppose that $m > n+1$ and that the result holds for all such families of fewer than m sets. It follows immediately that

$$\forall\, i \in \{1, \dots, m\}\ \exists\, x_i \in S_1 \cap \cdots \cap S_{i-1} \cap S_{i+1} \cap \cdots \cap S_m.$$

Then by application of Radon's theorem to the set $S = \{x_1, \dots, x_m\}$, there exist $C_1, C_2 \subseteq S$ such that $S = C_1 \sqcup C_2$ and $\operatorname{co}(C_1) \cap \operatorname{co}(C_2) \neq \emptyset$. Suppose *wlog* that

$$C_1 = \{x_1, \dots, x_k\} \qquad \text{and} \qquad C_2 = \{x_{k+1}, \dots, x_m\}$$

for some $k \in \{1, \dots, m-1\}$. Now observe that

$$\begin{aligned} &\forall\, i \in \{1, \dots, k\}\ x_i \in S_{k+1} \cap \cdots \cap S_m, \text{ and} \\ &\forall\, i \in \{k+1, \dots, m\}\ x_i \in S_1 \cap \cdots \cap S_k. \end{aligned}$$

Given this, and by the above principles, we have that

$$\begin{aligned} &\operatorname{co}(x_1, \dots, x_k) \subseteq S_{k+1} \cap \cdots \cap S_m, \text{ and} \\ &\operatorname{co}(x_{k+1}, \dots, x_m) \subseteq S_1 \cap \cdots \cap S_k. \end{aligned}$$

Then

$$\begin{aligned} \bigcap \mathscr{F} &= S_1 \cap \cdots \cap S_k \cap S_{k+1} \cap \cdots \cap S_m \\ &\supseteq \operatorname{co}(x_1, \dots, x_k) \cap \operatorname{co}(x_{k+1}, \dots, x_m) \\ &= \operatorname{co}(C_1) \cap \operatorname{co}(C_2) \neq \emptyset, \end{aligned}$$

and we are done. □

Lemma. *A set $S \subseteq \mathbb{R}^n$ is compact if, and only if, every family of closed subsets of S with the finite intersection property has nonempty intersection.*

Theorem (Helly)**.** *Let $\mathscr{F} = \{S_i\}_{i \in \mathcal{I}}$ be a family of compact, convex sets in $\mathbb{R}^n$. If every subfamily of $n+1$ sets in $\mathscr{F}$ has nonempty intersection, then $\bigcap \mathscr{F} \neq \emptyset$.*

Proof. Choose some $k \in \mathcal{I}$ arbitrarily such that $S_k \in \mathscr{F}$. Consider the family

$$\mathscr{G} := \mathscr{F} \cap S_k = \{S_i \cap S_k \mid i \in \mathcal{I}\}.$$

Let $\mathscr{H} \subseteq \mathscr{G}$ be a finite subfamily. Then there exists some finite $\mathcal{J} \subseteq \mathcal{I}$ such that $\mathscr{H} = \{S_j \cap S_k \mid j \in \mathcal{J}\}$. By the assumption and Helly's theorem,

$$\bigcap \mathscr{H} = \bigcap \{S_j \cap S_k \mid j \in \mathcal{J}\} = \bigcap \{S_j \mid j \in \mathcal{J} \cap \{k\}\} \neq \emptyset.$$

So $\mathscr{G}$ is a family of closed (by the Hahn-Banach theorem) subsets of S_k with the finite intersection property, and therefore by the above Lemma it follows that $\bigcap \mathscr{G} \neq \emptyset$. Then

$$\bigcap \mathscr{F} = \left(\bigcap \mathscr{F}\right) \cap S_k = \bigcap (\mathscr{F} \cap S_k) = \bigcap \mathscr{G} \neq \emptyset. \qquad \square$$

Proof of the claim

Theorem (Jung)**.** *Let $S \subseteq \mathbb{R}^n$. Then S is contained in a closed ball of radius $r \leq d\sqrt{\frac{n}{2(n+1)}}$, where $d := \operatorname{diam}(S)$.*

Proof. We split the proof into two cases, depending on the cardinality of S.

($|S| \leq n+1$). This is mostly a geometric argument.

($|S| > n+1$). Set $\rho := d\sqrt{\frac{n}{2(n+1)}}$ and consider for each $x \in S$ the closed ball $\overline{B}_\rho(x)$. Let $S' \subseteq S$ contain $n+1$ points and note that the previous case now applies. Then there exists a $z \in \mathbb{R}^n$ such that

$$S' \subseteq \overline{B}_\rho(z) \iff z \in \bigcap_{x \in S'} \overline{B}_\rho(x).$$

Applying Helly's theorem to the family $\mathscr{F} := \{\overline{B}_\rho(x) \mid x \in S\}$, we conclude that

$$\bigcap_{x \in S} \overline{B}_\rho(x) \neq \emptyset \iff \exists\, z \in \mathbb{R}^n \,:\, S \subseteq \overline{B}_\rho(z),$$

whence the result. □

Setting $d = 1$ and $n = 2$, the above claim now follows immediately from this result.