The Nash problem

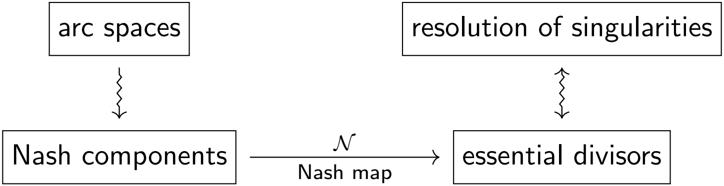
Alvin Šipraga

28 August 2015

Overview

1. resolution of singularities
2. essential divisors
3. arc spaces
4. Nash $\begin{cases} \text{components} \\ \text{map} \\ \text{problem} \end{cases}$
5. solution for toric varieties

Overview



John F. Nash, Jr., *Arc structure of singularities*, Duke Math. J. **81** (1995)

Resolution of singularities

X — singular variety over an algebraically closed field k

Idea

- "parametrise" the variety X with a smooth variety Y

Problems

- existence ($\operatorname{char} X = 0$, surfaces, toric varieties)
- no obvious choice

Approach

- classification
- minimal resolutions

Essential divisors

$f : Y \to X$ — resolution of singularities of X

Definition

- *prime divisor on* Y — closed subvariety of Y of codimension 1
- *exceptional divisor of* f — prime divisor E on Y such that $f(E)$ is of codimension ≥ 2

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Definition

- *exceptional divisor over* X — equivalence class of exceptional divisors of all resolutions of X
- *essential divisors over* Y — exceptional divisors over Y corresponding to irreducible components of $f^{-1}(\mathrm{Sing}\, X)$ for every resolution $f : Y \to X$

Essential divisors

$$
\begin{array}{ccc}
\{\text{prime divisors on } Y\} & & \\
\cup\!\mid & & \\
\{\text{exceptional divisors of } f\} & \overset{\sim}{\hookrightarrow} & \{\text{exceptional divisors over } X\} \\
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Arc spaces

X — scheme of finite type over an algebraically closed field k
K — field extension of k

Definition

- *arc on* X — morphism of the form

$$\operatorname{Spec} K[[t]] \to X$$

- *arc space of* X — scheme X_∞ whose K-valued points correspond to arcs on X

Arc spaces

Proposition

If $f : Y \to X$ is a resolution, then f_∞ induces a bijection

$$Y_\infty \setminus (f^{-1}(\operatorname{Sing} X))_\infty \cong X_\infty \setminus (\operatorname{Sing} X)_\infty.$$

Proposition

If X is a smooth scheme and $Z \subseteq X$ is an irreducible subscheme, then $\pi_X^{-1}(Z)$ is irreducible.

Nash components

X — singular variety

Definition

- *Nash component with respect to* X — irreducible component of $\pi_X^{-1}(\operatorname{Sing} X)$ containing at least one arc α such that $\alpha(\eta) \notin \operatorname{Sing} X$

Nash map

$f : Y \to X$ — arbitrary resolution of singularities
$\{C_i\}_{i \in \mathcal{I}}$ — Nash components (with respect to X)
$\{E_j\}_{j=1}^m$ — irreducible components of $f^{-1}(\operatorname{Sing} X)$

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$$\mathcal{N} : \{\text{Nash components}\} \to \{\text{irred. components of } f^{-1}(\mathrm{Sing}\, X)\}$$

Rule

$\mathcal{N}(C_i) = E_j$ means f_∞ maps the generic point of $\pi_Y^{-1}(E_j)$ to the generic point of C_i.

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Theorem (Nash)

The map $\mathcal{N}$ is injective onto the set of essential components over Y.

Nash problem

Is the Nash map $\mathcal{N}$ bijective?

Nash problem

Some answers to the problem:

Nash problem

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- 2003 — Ishii–Kollár
  - toric varieties — *yes*
  - dimension ≥ 4 — *no*

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  - surfaces — *yes*

Nash problem

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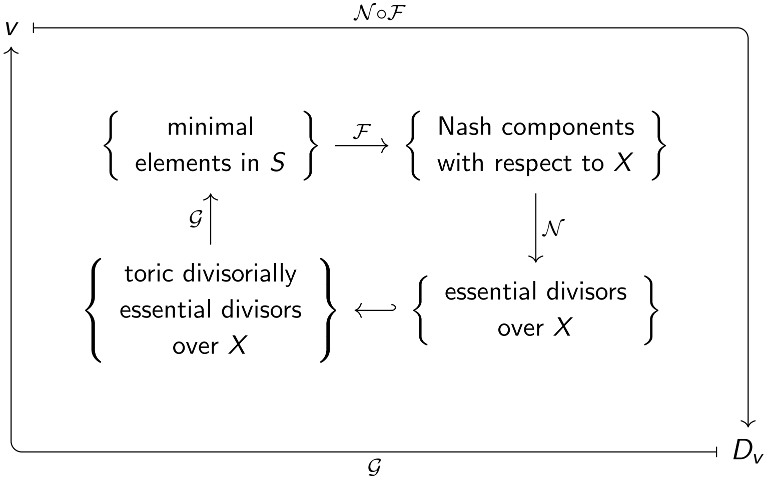
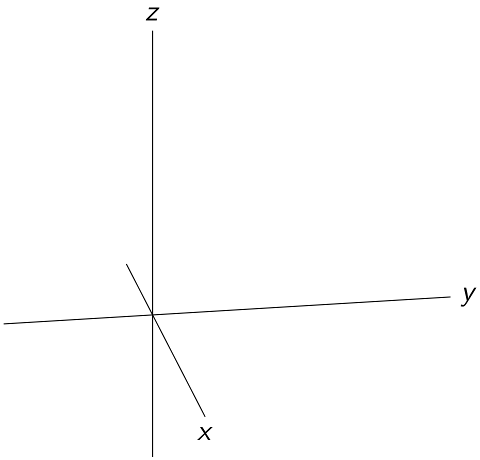
- 2003 — Ishii–Kollár
  - toric varieties — *yes*
  - dimension ≥ 4 — *no*
- 2012 — de Bobadilla–Pe Pereira
  - surfaces — *yes*
- 2013 — de Fernex
  - dimension ≥ 3 — *no*

The Nash problem for toric varieties

Shihoko Ishii and János Kollár, *The Nash problem on arc families of singularities*, Duke Math. J. **120** (2003)



Example toric variety



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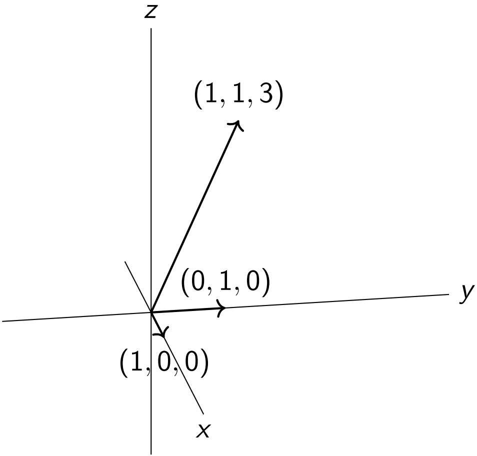

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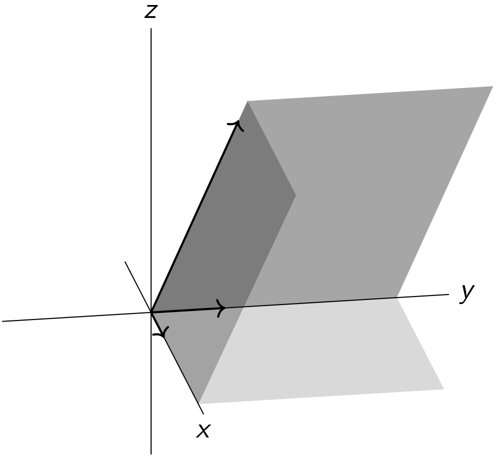

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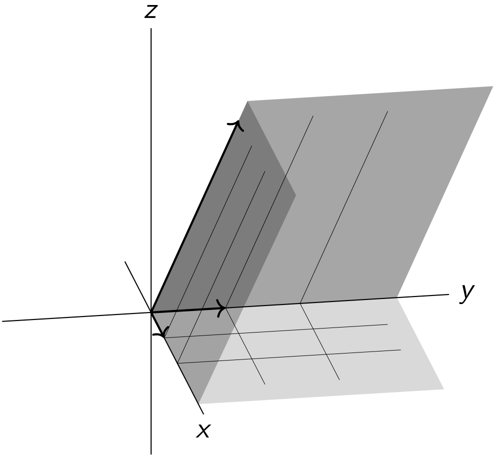

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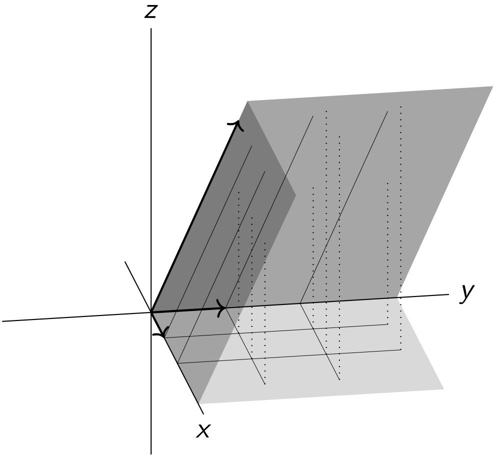

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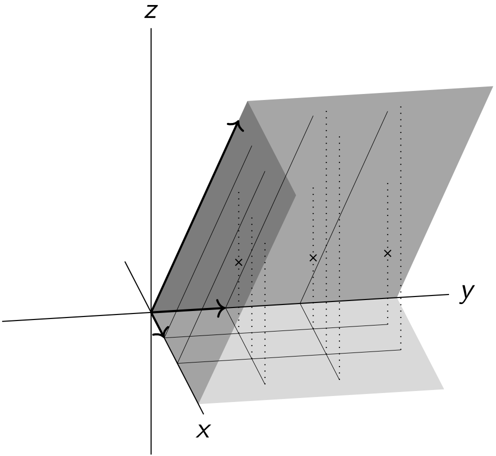

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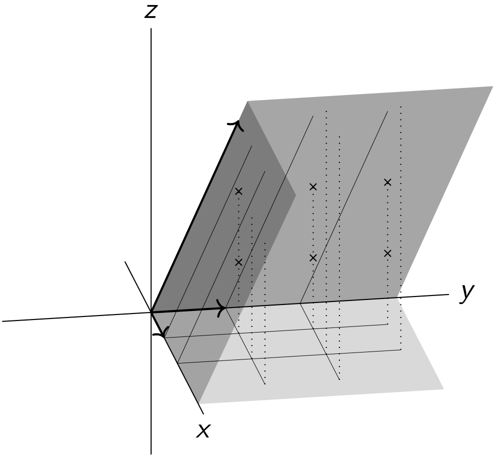

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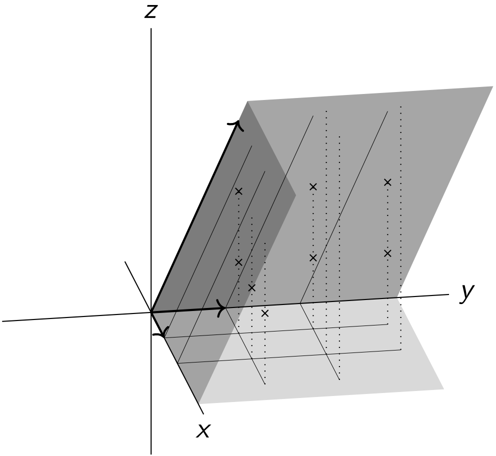

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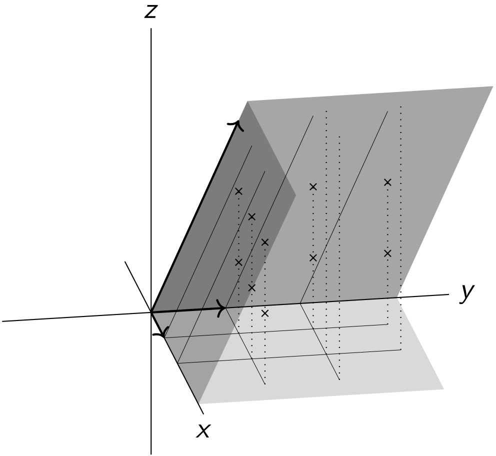

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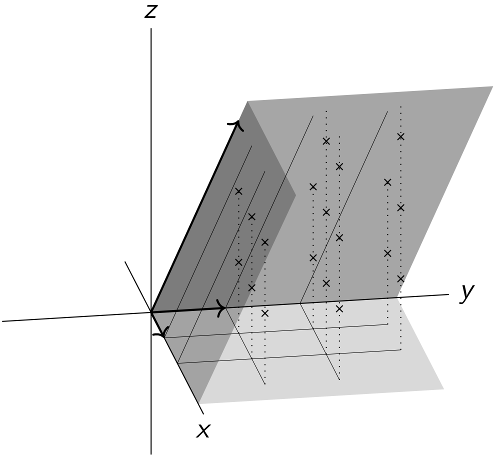

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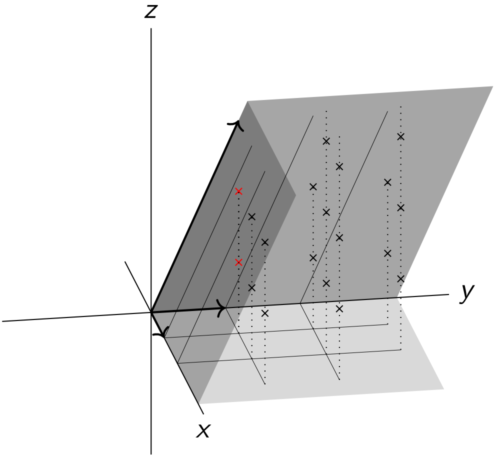

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